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Some Remarks on the Exact Controlled Approximation Order of Bivariate Splines on a Diagonal Mesh

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INTRODUCTION

In a fundamental series of papers de Boor and Höllig [2–4] and de Boor and De Vore [1] have introduced box splines and have shown their usefullness in multivariate approximation. They have answered many questions and have left several open ones.

One question de Boor and Höllig leave open is: What is the best degree of approximation obtainable? To state this question more precisely we introduce the following notation.

Let Δ be the mesh obtained from the lines $x = m$, $y = n$, $y = x + p$, where m, n, and p are arbitrary integers and $\Pi^{\rho}_{k,\Delta} = \Pi_{k,\Delta} \cap C^{\rho}$, the space of bivariate piecewise polynomial functions of total degree $\leqslant k$, on the mesh Δ, belonging to C^{ρ} globally. Set

$$m(k) = \min\{2(k-\rho), k+1\}.$$

de Boor and Höllig have shown that if $\Pi^{\rho}_{k,\Delta}$ has approximation order m, i.e.,

$$\operatorname{dist}(f, S_h(\Phi)) = O(h^m)$$

for all sufficiently smooth functions while

$$\operatorname{dist}(f, S_h(\Phi)) \neq O(h^{m+1}),$$

then $m \leqslant m(k)$. See below for definition of $S_h(\Phi)$. Further deBoor and Höllig:

Conjecture. The exact approximation order of $\Pi^{\rho}_{k,\Delta}$ never differs from its upper bound $m(k)$ by more than 1.

Recently, Jia [7] showed that the exact approximation order for $\Pi^{\rho}_{k,\Delta}$ never differs from $m(k)$ by more than 2.

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In this paper we do two things: (1) In view of the importance of Jia's result we give a simplification of his proof, using the techniques of Dahmen and Micchelli [5, 6]. (2) By using the notion of *controlled approximation* [6], we show that (in a sense to be explained below) Jia's results are sharp, see Theorem 1.

To this end, we quote Dahmen and Micchelli [6], who defined the notion of controlled approximation in any s-dimensional space in the following manner:

For any set $\Phi = \{\phi_1, ..., \phi_N\}$ of functions on $\mathbb{R}^s$ we define

$$S_h(\Phi) = \operatorname{span} \left\{ \phi_j \left(\frac{\cdot}{h} - \alpha \right) : \alpha \in Z^s, j = 1, ..., N \right\}, \qquad S_1(\Phi) = S(\Phi).$$

$S(\Phi)$ is said to admit *controlled L_p-approximation of order m, or* briefly

$$S(\Phi) \in A_{m,p},$$

if and only if for any $f \in C^\infty \cap W_p^m(\mathbb{R}^s)$ there exist weights $\omega_{j,\alpha}^h$ such that for any domain $\Omega \subset \mathbb{R}^s$ the inequalities

$$\left\| f - \sum_{j=1}^{N} \sum_{\alpha \in Z^s} \omega_{j,\alpha}^h \left(h^{-s/p} \phi_j \left(\frac{\cdot}{h} - \alpha \right) \right) \right\|_p (\Omega) \leqslant C h^m \, |f|_{p,m} (\Omega_{rh})$$

while

$$\|\{\omega_{j,\alpha}^h\}\|_{l_p}(\Omega) := \left(\sum_{j,\alpha} |\omega_{j,\alpha}^h|^p \right)^{1/p} \leqslant C \, \|f\|_p \, (\Omega_{rh})$$

$$\operatorname{supp} \left(\phi_j \left(\frac{\cdot}{h} - \alpha \right) \right) \cap \Omega \neq \varnothing$$

hold for some constants C, r independent of h, Ω, f. Here

$$W_p^m(\Omega) := \{ f : |f|_{p,m}(\Omega) := \left(\sum_{|\beta| = m} \|D^\beta f\|_p^p(\Omega) \right)^{1/p} < \infty \}$$

denotes the usual Sobolev spaces and $\Omega_h = \{x : \operatorname{dist}_2(x, \Omega) \leqslant h\}$.

Our set Φ will be the set of all box splines φ_i belonging to $\Pi_{k,\Delta}^\rho$. This is defined as follows:

Let e_i be the unit vector along the ith axis ($i = 1, 2$) and

$$d_1 := e_1, \qquad d_2 := e_1 + e_2, \qquad d_3 := e_2.$$

For some positive integers r, s, t let

$$q_1 = q_2 = \cdots = q_r = d_1; \qquad q_{r+1} = \cdots = q_{r+s} = d_2;$$

and

$$q_{r+s+1} = \cdots = q_{r+s+t} = d_3.$$

As defined in [2], the box spline $M_{r,s,t}$ is the distribution in R^2 given by the rule:

$$\langle M_{r,s,t}, \varphi\rangle = \int_{[0,1]^{r+s+t}} \varphi\left(\sum \lambda_i q_i\right) d\lambda; \varphi \in C_0^\infty(R^2).$$

As shown in [2] and [4], $M_{r,s,t} \in \Pi_{k,\Delta}^\rho$, where

$$\begin{aligned} &k = n-2, \qquad n = r+s+t, \qquad \rho = d-1 \\ &d = \min(r+s, r+t, s+t) - 1 = n - 1 - \max(r, s, t). \end{aligned} \tag{1}$$

Further

$$\hat{M}_{r,s,t}(\xi_1, \xi_2) = (\rho(\xi_1))^r\, (\rho(\xi_1+\xi_2))^s\, (\rho(\xi_2))^t$$

with $\hat{f}$ the Fourier transform of f, and $\rho(\xi) = (1 - e^{-i\xi})/i\xi$.

If Φ is the set of all box splines belonging to $\Pi_{k,\Delta}^\rho$ and if $S(\Phi) \in A_{m,\infty}$ but $S(\Phi) \notin A_{m+1,\infty}$ we will say $\Pi_{k,\Delta}^\rho$ has exact controlled approximation order m.

It is important to note that this terminology does not presuppose that $\Pi_{k,\Delta}^\rho$ may have a *different* order of exact controlled approximation relative to another set of functions in $\Pi_{k,\Delta}^\rho$ which are different from box splines.

THEOREM 1. *The exact controlled approximation order of $\Pi_{7,\Delta}^3$ is* 6.

Remark 1. For $k=4$ the maximal smoothness for which the space $\Pi_{k,\Delta}^\rho$ is dense as $|\Delta| \to 0$ is $\rho = 4$.

We consider the less smooth space $S = \Pi_{7,\Delta}^3$. Here $2(k-\rho) = k+1 = m(k) = 8$. We will show its exact controlled approximation order is $6 = m(k) - 2$.

Proof. For future reference we list the lowest order terms in the power series expansion about $(2\pi m, 2\pi n)$ with variables $\xi_1 = 2\pi m + \delta_1$, $\xi_2 = 2\pi n + \delta_2$ for $\hat{M}_{r,s,t}(\xi_1, \xi_2)$

$$\begin{aligned} \hat{M}_{r,s,t}(\xi_1, \xi_2) &= 1, && m=n=0 \\ &= [\rho'(2\pi m)]^r\, [\rho'(2\pi(n+m))]^s \\ &\quad \times [\rho'(2\pi n)]^t\, \delta_1^r(\delta_1+\delta_2)^s\, \delta_2^t, && m\neq 0, n\neq 0, m\neq -n \\ &= [\rho'(2\pi m)]^{r+s}\, \delta_1^r(\delta_1+\delta_2)^s, && m\neq 0, n=0 \\ &= [\rho'(2\pi n)]^{s+t}\, (\delta_1+\delta_2)^s\, \delta_2^t, && m=0, n\neq 0 \\ &= (-1)^t\, [\rho'(2\pi m)]^{r+t}\, \delta_1^r \delta_2^t, && m\neq 0, n\neq 0, m=-n. \end{aligned} \tag{2}$$

Further note $\rho'(2\pi p) = 1/2\pi p$, p an integer $\neq 0$.

From (1) it is easily checked that the only box splines that are in $\pi^3_{7,\Delta}$ are the following 13 splines. $\Phi = \{\varphi_1, ..., \varphi_{13}\}$, where

$$\begin{array}{llll} \varphi_1 = M_{2,3,4} & \varphi_2 = M_{2,4,3} & \varphi_3 = M_{3,2,4} & \varphi_4 = M_{3,4,2} \\ \varphi_5 = M_{4,2,3} & \varphi_6 = M_{4,3,2} & \varphi_7 = M_{1,4,4} & \varphi_8 = M_{4,1,4} \\ \varphi_9 = M_{4,4,1} & \varphi_{10} = M_{2,3,3} & \varphi_{11} = M_{3,2,3} & \varphi_{12} = M_{3,3,2} \\ \varphi_{13} = M_{3,3,3}. & & & \end{array}$$

From (2) it follows that:

$$D^\beta \hat{\varphi}_i(2\pi\alpha) = 0 \qquad \text{for} \quad \alpha \in Z^2 - \{0\}, |\beta| \leqslant 4, i = 1, ..., 13. \tag{3}$$

(Equation (3) follows in the general case for $|\beta| \leqslant d$; see [5].)

Let ψ_0 be a linear combination of the above φ_i, $i = 1, ..., 13$.

In [1] it is shown that

THEOREM 2. *A necessary condition that $\pi^3_{7,\Delta}$ be of exact controlled approximation order 7 is that there exists ψ_0 so that*

$$D^\beta \hat{\psi}_0(2\pi\alpha) = 0, \qquad \alpha \in Z^2 - \{0\}, \qquad |\beta| = 5, \qquad \hat{\psi}_0(0, 0) = 1 \tag{4}$$

and there exist constant $a_{i\gamma\beta}$ such that

$$D^\beta \hat{\psi}_0(2\pi\alpha) + \sum_{\substack{|\gamma| = 5 \\ i = 1,13}} a_{i\gamma\beta} \hat{\varphi}_i(2\pi\alpha) = 0, \qquad |\beta| = 6 \tag{5}$$

We now show that there does not exist ψ_0 satisfying both (4) and (5) (see also Dahmen and Micchelli [6, Proof of Theorem 3.1 especially the case $l = 1$]).

Say $\psi_0 = \sum_{i=1}^{13} a_i \varphi_i$.

Using (2) we list the lowest order terms in the power series expansion about $(2\pi m, 2\pi n)$ with variables $\xi_1 = 2\pi m + \delta_1$, $\xi_2 = 2\pi n + \delta_2$ for $\hat{\psi}_0$.

$$\begin{aligned} \hat{\psi}_0(\xi_1, \xi_2) &= \sum_{i=1}^{13} a_i, \qquad m = n = 0; \\ &= O(\delta_1^q \delta_2^p), \qquad q + p = 8, p \geqslant 1, q \geqslant 1, m \neq -n, m \neq 0, n \neq 0 \\ &\quad \text{(or more generally } q + p = r + s + t - 1) \\ &= [\rho'(2\pi m)]^5 \, [(a_1 + a_{10})\, \delta_1^2(\delta_1 + \delta_2)^3 + (a_3 + a_{11})\, \delta_1^3(\delta_1 + \delta_2)^2 \\ &\quad + a_7 \delta_1(\delta_1 + \delta_2)^4 + a_8 \delta_1^4(\delta_1 + \delta_2)], \qquad m \neq 0, n = 0; \end{aligned} \tag{6}$$

$$= [\rho'(2\pi n)]^5 \, [(a_5 + a_{11})(\delta_1 + \delta_2)^2 \, \delta_2^3 + (a_6 + a_{12})(\delta_1 + \delta_2)^3 \, \delta_2^2$$
$$+ a_8(\delta_1 + \delta_2) \, \delta_2^4 + a_9(\delta_1 + \delta_2)^4 \, \delta_2], \qquad m = 0, n \neq 0$$
$$= [\rho'(2\pi m)]^5 \, [-(a_2 + a_{10}) \, \delta_1^2 \delta_2^3 + (a_4 + a_{12}) \, \delta_1^3 \delta_2^2$$
$$+ a_7 \delta_1 \delta_2^4 - a_9 \delta_1^4 \delta_2], \qquad m = -n \neq 0.$$

These equations imply that for $\hat{\psi}_0$ to satisfy (4), one needs:

$$a_1 + a_{10} = a_3 + a_{11} = a_5 + a_{11} = a_6 + a_{12} = a_2 + a_{10}$$
$$= a_4 + a_{12} = a_7 = a_8 = a_9 = 0 \tag{7a}$$
$$\sum_{i=1}^{13} a_i = 1. \tag{7b}$$

Assuming $\hat{\psi}_0$ satisfies (7a) and noting that $[\rho(\delta_2)]^4 - [\rho(\delta_2)]^3 = \rho'(0) \, \delta_2 +$ higher order terms, we list the new lowest order terms of $\hat{\psi}_0$ in the power series expansion about $(2\pi m, 2\pi n)$ with variables $\xi_1 = 2\pi m + \delta_1, \xi_2 = 2\pi n + \delta_2$

$$\hat{\psi}_0(\xi_1, \xi_2)$$
$$= [\rho'(2\pi m)]^6 \, [a_2 \delta_1^2 (\delta_1 + \delta_2)^4 + a_5 \delta_1^4 (\delta_1 + \delta_2)^2$$
$$+ (a_{12} + a_{13}) \, \delta_1^3 (\delta_1 + \delta_2)^3]$$
$$+ [\rho'(2\pi m)]^5 \, [\rho'(0)][a_1 \delta_1^2 (\delta_1 + \delta_2)^3 \, \delta_2 + a_3 \delta_1^3 (\delta_1 + \delta_2)^2 \, \delta_2],$$
$$m \neq 0, n = 0 \tag{3a}$$

$$\hat{\psi}_0(\xi_1, \xi_2)$$
$$= [\rho'(2\pi n)]^6 \, [a_3 (\delta_1 + \delta_2)^2 \, \delta_2^4 + a_4 (\delta_1 + \delta_2)^4 \, \delta_2^2$$
$$+ (a_{10} + a_{13})(\delta_1 + \delta_2)^3 \, \delta_2^3]$$
$$+ [\rho'(2\pi n)]^5 \, [\rho'(0)][a_5 (\delta_1 + \delta_2)^2 \, \delta_2^3 \delta_1 + a_6 \delta_1 (\delta_1 + \delta_2)^3 \, \delta_2^2],$$
$$m = 0, n \neq 0. \tag{8b}$$

From our derivation of (6) it is clear that $D^\gamma \hat{\varphi}_i(2\pi a)$, $|\gamma| = 5$, will have no factor $[\rho'(2\pi m)]^6$ at $(2\pi m, 0)$, $m \neq 0$.

Hence to satisfy (5), all terms in (8a) that have $[\rho'(2\pi m)]^6$ as a factor must be zero. This implies

$$a_2 = a_5 = a_{12} + a_{13} = 0. \tag{9a}$$

Similarly we find applying the analogous argument to (8b) that

$$a_3 = a_4 = a_{10} + a_{13} = 0. \tag{9b}$$

But (9a), (9b) and (7a) imply $a_i = 0$, $i = 1,..., 13$, hence (7b) cannot be satisfied. Thus we have shown that there is no ψ_0 satisfying (4) and (5), hence $\pi^3_{7,\Delta}$ is not of exact controlled approximation order 7.

Comments on Jia's Theorem

The key to Jia's [7] results, is his following theorem:

THEOREM 3. *If $k = 2\rho + 2$ and $S = \pi^\rho_{k,\Delta}$, then the exact controlled approximation order m of $\Pi^\rho_{k,\Delta}$ satisfies $m \geqslant k$*

In this section of the paper, we will give a simplified proof of this theorem.

To prove Theorem 3, we will use the following criteria (see Dahmen and Micchelli [5], Strang and Fix [9, Theorem 1]).

THEOREM 4. *Let $\hat{B}$ be the Fourier transform of a function B which is of compact support and which belongs to $\Pi^\rho_{k,\Delta}$. If*

$$\begin{gathered} D^\beta \hat{B}(2\pi\alpha) = 0, \qquad \alpha \in Z^2 - \{0\}, \qquad |\beta| \leqslant q \\ \hat{B}(0) \neq 0 \end{gathered} \tag{10}$$

then it follows that the exact controlled approximation order m of $\Pi^\rho_{k,\Delta}$ satisfies $m \geqslant q + 1$

For our proof we will also need the following lemma.

LEMMA 1. *For k an integer, there exists a trigonometric polynomial*

$$P_k(\delta) = \sum_{l=0}^{k-2} b_{l,k} e^{i\delta l} \tag{11}$$

so that

$$P_k(\delta) \times \left(\frac{1 - e^{-i\delta}}{i\delta}\right)^k = 1 + O(\delta^{k-1}) \qquad \text{for small } \delta. \tag{12}$$

Proof. Let $f_k(\delta) = (i\delta/1 - e^{-i\delta})^k$ for small δ. Then determine $b_{l,k}$ so that

$$P_k^{(\alpha)}(0) = f_k^{(\alpha)}(0), \qquad \alpha = 0, 1,..., k-2. \tag{13}$$

It is easily seen that the linear equation (13) to determine the $b_{l,k}$ has the Van der Monde determinant, so (13) can be solved. Then since $P_k(\delta) = (i\delta/1 - e^{-i\delta})^k + O(\delta^{k-1})$, (12) follows.

PROOF OF JIA'S THEOREM

Following Jia, instead of considering $M_{r,s,t}$ we consider a finite sum of its translates. In our case we consider one whose Fourier transform is

$$\hat{B}_{r,s,t}(\xi_1, \xi_2) = P_r(\xi_1)\, P_s(\xi_1 + \xi_2)\, P_t(\xi_2)\, \hat{M}_{r,s,t}(\xi_1, \xi_2)$$

and instead of one triplet of numbers (r, s, t) consider all numbers from the sets

$$\begin{aligned} I &= \{(r, s, t) | r + s + t = 2\rho + 4 \text{ and } 2 \leqslant r, s, t \leqslant \rho + 1\} \\ J_1 &= \{(r, s, t) | r + s + t = 2\rho + 3 \text{ and } 2 \leqslant r, s, t \leqslant \rho + 1\} \\ J_2 &= \{(r, s, t) | r + s + t = 2\rho + 3 \text{ and } 2 \leqslant r, s, t \leqslant \rho\} \\ K &= \{(r, s, t) | r + s + t = 2\rho + 2 \text{ and } 2 \leqslant r, s, t \leqslant \rho\}. \end{aligned} \tag{14}$$

Then define

$$B := \sum_{(r,s,t)\in I} B_{r,s,t} - \sum_{(r,s,t)\in J_1} B_{r,s,t} - \sum_{(r,s,t)\in J_2} B_{r,s,t} + \sum_{(r,s,t)\in K} B_{r,s,t}.$$

From (1) it follows that $B \in \Pi^{\rho}_{2\rho+2,\Delta}$. We now verify that $\hat{B}$ satisfies the assumptions of Theorem 4 for $q = k - 1$. From (2), it is clear that (10) is satisfied for $\alpha = (m, n)$ and $m \neq n$, $n \neq 0$, $m + n \neq 0$. Let us now consider the case when $m \neq 0$, $n \neq 0$, $m + n = 0$. For this purpose write I, J_1, J_2, K in the form

$$\begin{aligned} I &= \{\rho + 1, 2, \rho + 1\} \\ &\quad \cup \{(r, s, t) |\, 3 \leqslant s \leqslant \rho + 1, r + t = 2\rho + 4 - s, 2 \leqslant r, t \leqslant \rho + 1\} \\ J_2 &= \{(r, s, t) | 3 \leqslant s \leqslant \rho, r + t = 2\rho + 3 - s, 2 \leqslant r, t \leqslant \rho\} \\ J_1 &= \{(r, s, t) | 2 \leqslant s \leqslant \rho, r + t = 2\rho + 3 - s, 2 \leqslant r, t \leqslant \rho + 1\} \\ &\quad \cup \{(r, s, t) | s = \rho + 1, r + t = \rho + 2, 2 \leqslant r, t \leqslant \rho\} \\ K &= \{(r, s, t) | 2 \leqslant s \leqslant \rho - 1, r + t = 2\rho + 2 - s,\ 2 \leqslant r, t \leqslant \rho\} \\ &\quad \cup \{(r, s, t) | s = \rho,\ r + t = \rho + 2, 2 \leqslant r, t \leqslant \rho\}. \end{aligned} \tag{15}$$

Then

$$\hat{B} = \hat{B}_{\rho+1,2,\rho+1} + \sum_{\substack{3 \leqslant s \leqslant \rho+1 \\ r+t=2\rho+4-s \\ 2 \leqslant r,t \leqslant \rho+1}} (\hat{B}_{r,s,t} - \hat{B}_{r,s-1,t})$$
$$- \sum_{\substack{3 \leqslant s \leqslant \rho \\ r+t=2\rho+3-s \\ 2 \leqslant r,t \leqslant \rho}} (\hat{B}_{r,s,t} - \hat{B}_{r,s-1,t}) - \sum_{\substack{s=\rho+1 \\ r+t=\rho+2 \\ 2 \leqslant r,t \leqslant \rho}} (\hat{B}_{r,s,t} - \hat{B}_{r,s-1,t}), \tag{16}$$

or less precisely as

$$\hat{B} = \hat{B}_{\rho+1,2,\rho+1} \pm \sum (\hat{B}_{r,s,t} - \hat{B}_{r,s-1,t}). \tag{17}$$

From (2) we find for $\xi_1 = 2\pi m + \delta_1$, $\xi_2 = 2\pi n + \delta_2$, $m \neq 0$, $n \neq 0$, $m + n = 0$.

$$\hat{B}_{\rho+1,2,\rho+1} = O(\delta_1^{\rho+1} \delta_2^{\rho+1}) \tag{18}$$

and from (2) and (13)

$$\begin{aligned} &\hat{B}_{r,s,t} - \hat{B}_{r,s-1,t} \\ &\quad = P_r(2\pi m + \delta_1) P_t(-2\pi m + \delta_2)[\rho(2\pi m + \delta_1)]^r [\rho(-2\pi m + \delta_2)]^t \\ &\qquad \times [P_s(\delta_1 + \delta_2)[\rho(\delta_1 + \delta_2)]^s \\ &\qquad - P_{s-1}(\delta_1 + \delta_2)[\rho(\delta_1 + \delta_2)]^{s-1}] \\ &\quad = O(\delta_1^r \delta_2^t)[O(\delta_1 + \delta_2)^{s-1} - O(\delta_1 + \delta_2)^{s-2}] \\ &\quad = O(\delta_1^r \delta_2^t (\delta_1 + \delta_2)^{s-2}). \end{aligned} \tag{19}$$

Thus (10) is satisfied at $m \neq 0$, $n \neq 0$, $m + m = 0$. By similar reasoning (10) is also verified at all other points $2\pi\alpha$ $\alpha \in Z^2 - \{0\}$. For example, to treat the points $m = 0$, $n \neq 0$, we note that r, s, t appear symmetrically in I, J_1, J_2 and K in (14). Thus instead of developing $\hat{B}$ in the form (17), we can just as well develop it as

$$\hat{B} = \hat{B}_{2,\rho+1,\rho+1} \pm \sum (\hat{B}_{r,s,t} - \hat{B}_{r-1,s,t}) \tag{20}$$

Proceeding as in (18) and (19) we find that at $m = 0$, $n \neq 0$

$$B_{2,\rho+1,\rho+1} = (\delta_1 + \delta_2)^{\rho+1} \delta_2^{\rho+1}$$
$$B_{r,s,t} - B_{r-1,s,t} = O((\delta_1 + \delta_2)^s \delta_2^t \delta_1^{r-2}).$$

Hence we see that (10) is also verified at $m = 0$, $n \neq 0$. Analogous reasoning establishes the result at $n = 0$, $m \neq 0$ and completes the proof.

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